J. Fluid Mech. (1997), *vol.* 344, *pp.* 335–337. Printed in the United Kingdom

The effective film viscosity coefficients of a thin floating fluid layer

By ALASTAIR D. JENKINS[1] AND KRISTIAN B. DYSTHE[2]

[1]Nansen Environmental and Remote Sensing Center, Edv. Griegs vei 3a, N-5037 Solheimsviken, Bergen, Norway

[2]Mathematical Institute, University of Bergen, Joh. Bruns gt. 12, N-5008 Bergen, Norway

(Received 28 November 1995 and in revised form 7 April 1997)

We derive a constitutive relation, relating the tangential stress, tangential velocity, thickness h, and viscosity μ, for a thin layer of Newtonian fluid on top of a fluid substrate. We find that the upper layer exerts a viscous tangential shear stress on the lower fluid, behaving as if it were a film with a two-dimensional shear viscosity equal to μh, and a dilatational viscosity $3\mu h$.

For a surface film on a substrate of water, a constitutive relation (cf. Dorrestein 1951; Goodrich 1961; Miles 1967) is sometimes adopted between the tangential stress $\boldsymbol{T}$, the surface tension σ, and the tangential surface velocity $\boldsymbol{U}$:

$$\boldsymbol{T} = \nabla_h \sigma + \eta_1^s \nabla_h(\nabla_h \cdot \boldsymbol{U}) + \eta_2^s \nabla_h^2 \boldsymbol{U}. \tag{1}$$

Here, ∇_h and ∇_h^2 are the surface gradient and Laplacian operators, and η_1^s and η_2^s are the dilatational and shear viscosities of the film. (Dorrestein assumes that $\eta_1^s = 0$.)

The form (1) has mainly been used for very thin films or monolayers. The thickness of such a monolayer is typically of the order of the length of the molecules constituting the film, say 1–10 nm. Where optical interference phenomena (Newton's rings) are visible in a film, the thickness is, however, typically 500–1000 nm, i.e. 100–1000 times the molecular length. For interfaces between immiscible fluids and for monolayers the surface viscosities η_1^s and η_2^s do not seem to be simply related to the viscosities of the bulk fluids (cf. Edwards, Brenner & Wasan 1991 §16.3).

We wish to explore the case where the film is thick enough to behave like a Newtonian fluid, yet thin enough for the variation of the velocity field within the film to be reasonably small. In that case it is perhaps reasonable to assume that the surface viscosities mentioned above can be dominated by the viscous contribution of the bulk of the fluid layer. The coefficients $\eta_{1,2}^s$ are probably unrelated to the bulk viscosity of the film material, and their presence is due to molecular effects in a film whose thickness is of molecular dimension. We believe that $\eta_{1,2}^s \to 0$ as the film thickness increases, and in any case should become small in comparison with the viscous forces. Thus we assume

$$\eta_{1,2}^s / h \ll \mu,$$

where h is the thickness of the fluid layer and μ is its dynamic viscosity, and in the following we neglect $\eta_{1,2}^s$.

When periodic phenomena such as surface waves (of angular frequency ω, say) are studied, an oscillating boundary layer is created. We must assume that the thickness of the fluid layer is less than the thickness of the boundary layer, i.e. the upper limit of the film thickness h that we are concerned with should satisfy

$$h < \left(\frac{2\mu}{\omega\rho}\right)^{1/2}, \tag{2}$$

where ρ is the density of the film fluid.

We shall demonstrate that within the above limitations we obtain a constitutive relation of the type (1), but where η_1^s and η_2^s are replaced by $3\mu h$ and μh, respectively.

We employ a coordinate system (s, r, n), where s and r are orthogonal coordinates, parallel to the fluid surface, and n is directed upward, normal to the surface. The surface is at $n = 0$ and the interface is at $n = -h$. We assume that the fluid velocity $(\boldsymbol{u}, w) \equiv (u_s, u_r, w)$ varies slowly with n, so that it can be expressed as a Taylor expansion using coefficients depending only on s and r:

$$\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{u}_1 n + \boldsymbol{u}_2 n^2 + O(n^3), \tag{3a}$$

$$w = W - (\nabla_h \cdot \boldsymbol{U})n - \tfrac{1}{2}(\nabla_h \cdot \boldsymbol{u}_1)n^2 + O(n^3), \tag{3b}$$

where $\nabla_h \equiv (\partial/\partial s, \partial/\partial r, 0)$ and we have used the continuity equation $\nabla_h \cdot \boldsymbol{u} + \partial w/\partial n = 0$ to determine the expansion coefficients for w in terms of those for $\boldsymbol{u}$. We have also neglected terms of relative order κh, where κ is the mean curvature of the surface.

For $0 \leqslant n \leqslant h$, the viscous tangential stress $\boldsymbol{T}$ will be

$$\boldsymbol{T} = \mu\left(\frac{\partial \boldsymbol{u}}{\partial n} + \nabla_h w\right) \tag{4a}$$

$$= \mu[\boldsymbol{u}_1 + \nabla_h W + n(2\boldsymbol{u}_2 - \nabla_h(\nabla_h \cdot \boldsymbol{U}))], \tag{4b}$$

where we have neglected $O(n^2)$ terms. The corresponding normal stress will be

$$N = 2\mu\frac{\partial w}{\partial n} = 2\mu(-\nabla_h \cdot \boldsymbol{U} - n\nabla_h \cdot \boldsymbol{u}_1). \tag{5}$$

At the free surface ($n = 0$), we assume that we have no applied shear stress, so that $\boldsymbol{T} = 0$:

$$\boldsymbol{u}_1 + \nabla_h W = 0; \tag{6}$$

and the pressure p and normal stress N are related by $p - N = P - \sigma\kappa$, where P is the external applied constant pressure, σ is the surface tension, and κ is the mean curvature of the surface, so that

$$\nabla_h(P - \sigma\kappa) = \nabla_h(p - N) = \nabla_h p + 2\mu\nabla_h(\nabla_h \cdot \boldsymbol{U}) \quad \text{at } n = 0. \tag{7}$$

At the interface, (4b) and (6) give

$$\boldsymbol{T} = \mu h(\nabla_h(\nabla_h \cdot \boldsymbol{U}) - 2\boldsymbol{u}_2) \quad \text{at } n = -h. \tag{8}$$

The conservation of momentum parallel to the surface gives

$$\rho\left(\frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}t}\right)_h + \nabla_h p - \rho \boldsymbol{g}_h = \mu(\nabla_h^2 \boldsymbol{U} + 2\boldsymbol{u}_2) + O(n), \quad -h \leqslant n \leqslant 0, \tag{9}$$

where $(\mathrm{D}\boldsymbol{U}/\mathrm{D}t)_\mathrm{h}$ is the tangential fluid acceleration and $\boldsymbol{g}_\mathrm{h}$ is the acceleration due to gravity resolved along the coordinate plane tangential to the surface. To lowest significant order (i.e. neglecting the $O(n)$ terms), all the terms in (9) are independent of n. One can therefore eliminate $\nabla_h p$ between (7) and (9), obtaining

$$2\mu\boldsymbol{u}_2 = \rho\left(\frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}t}\right)_h - \rho\boldsymbol{g}_h - \mu\nabla_h^2\boldsymbol{U} - 2\mu\nabla_h(\nabla_h \cdot \boldsymbol{U}) + \nabla_h P - \sigma\nabla_h\kappa, \tag{10}$$

from which in the limit as $n \to -h$:

$$\boldsymbol{T} = \boldsymbol{T}_1 + \boldsymbol{T}_2 + O(h^2);$$
$$\boldsymbol{T}_1 = \mu h(3\nabla_h(\nabla_h \cdot \boldsymbol{U}) + \nabla_h^2 \boldsymbol{U}),$$
$$\boldsymbol{T}_2 = -h\left(\rho\left(\frac{\mathrm{D}U}{\mathrm{D}t}\right)_h - \rho \boldsymbol{g}_h + \nabla_h P - \sigma\nabla_h\kappa\right). \qquad (11)$$

Equation (11) gives the shear stress at the lower boundary of the upper fluid.

The second term $\boldsymbol{T}_2$ in (11) is independent of the viscosity μ of the upper fluid, so that the viscous tangential stress with which the fluid layer acts on the fluid below is given by the first term $\boldsymbol{T}_1$. Comparing (11) with (1), we thus see that the upper fluid layer behaves as if it were a film with a constitutive relation of type (1), with a two-dimensional shear viscosity μh and a two-dimensional dilatational viscosity $3\mu h$. There is also of course a normal force balance at the interface, but we do not need it to obtain this main result.

It is shown by Jenkins & Jacobs (1997) that provided that the upper fluid layer is sufficiently thin, the bulk viscosity of the upper fluid affects the damping of linear surface waves by effectively adding $4\mu h$ to the surface-film viscosity. In this case, the shear and dilatational surface viscosities are added together, since the wave-induced tangential surface motions are essentially one-dimensional (cf. Miles 1967).

The contribution of A. D. Jenkins was supported by the Research Council of Norway under the project 'Synthetic Aperture Radar Analysis and Modelling for Ocean Monitoring'. Constructive criticism from three anonymous referees helped to improve the manuscript.

REFERENCES

Dorrestein, R. 1951 General linearized theory of the effect of surface films on water ripples, I–II. *Proc. K. Ned. Akad. Wet.* B **54**, 260–272 & 350–356.

Edwards, D. A., Brenner, H. & Wasan, D. T. 1991 *Interfacial Transport Processes and Rheology*. Butterworth.

Goodrich, F. C. 1961 The mathematical theory of capillarity, I–III. *Proc. R. Soc. Lond.* A **260**, 481–489, 491–502 & 503–509.

Jenkins, A. D. & Jacobs, S. J. 1997 Wave damping by a thin layer of viscous fluid. *Phys. Fluids* **9**, 1256–1264.

Miles, J. W. 1967 Surface-wave damping in closed basins. *Proc. R. Soc. Lond.* A **297**, 459–475.